MATH 2020 Advanced Calculus II

Tutorial 4

1. Find the volume of the solid bounded by $z = 1 + x$, $z = 0$ and $r = \cos 2\theta$.

Solution. Notice that the curve $r = \cos 2\theta$ lies in $\{x \geqslant 0\}$ and is tangent to the two straight lines $\theta = \pm\frac{\pi}{4}$ at the origin, and hence the lower and upper limits for θ are $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ respectively. The volume is

$$
\begin{aligned}
& \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} z\, r dr d\theta \\
= & \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} (1 + x)\, r dr d\theta \\
= & \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos 2\theta} (1 + r\cos\theta)\, r dr d\theta \\
= & \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2} + \frac{r^3}{3}\cos\theta\right]_0^{\cos 2\theta} d\theta \\
= & \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{2}\cos^2 2\theta + \frac{1}{3}\cos^3 2\theta\cos\theta\right) d\theta \\
= & \left[\frac{1}{4}\left(\theta + \frac{1}{4}\sin 4\theta\right)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^3 \cos\theta d\theta \\
= & \frac{\pi}{8} + \frac{1}{3}\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1 - 6t^2 + 12t^4 - 8t^6) dt \\
= & \frac{\pi}{8} + \frac{2}{3\sqrt{2}}\left(1 - \frac{6}{3}\times\frac{1}{2} + \frac{12}{5}\times\frac{1}{2^2} - \frac{8}{7}\times\frac{1}{2^3}\right) \\
= & \frac{\pi}{8} + \frac{16\sqrt{2}}{105}.
\end{aligned}
$$

2. Find the volume of the solid bounded by $\rho = \frac{1}{2}$, $\rho = \cos\phi$ and $z = 0$.

Solution. Notice that the surfaces $\rho = \frac{1}{2}$ and $\rho = \cos\phi$ intersect along the circle $\{\rho = \frac{1}{2},\ \phi = \frac{\pi}{3}\}$. Thus the volume is given by

$$\begin{aligned}
&\int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_{\cos\phi}^{\frac{1}{2}} \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= 2\pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{1}{3}\rho^3\right]_{\cos\phi}^{\frac{1}{2}} \sin\phi \, d\phi \\
&= \frac{2\pi}{3}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{8} - \cos^3\phi\right) \sin\phi \, d\phi \\
&= \frac{2\pi}{3}\left[\frac{1}{8}(-\cos\phi) - \frac{1}{4}(-\cos^4\phi)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
&= \frac{2\pi}{3}\left[\frac{1}{8}\left(\frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{2}\right)^4\right] \\
&= \frac{\pi}{32}.
\end{aligned}$$

3. Find the average value of the function $f(x, y, z) = x^2 + y^2 = r^2$ over the unit ball $B = \{\rho \leqslant 1\}$.

Solution. The average of f is defined to be

$$\begin{aligned}
\frac{\iiint_B f \, dV}{\iiint_B dV} &= \frac{1}{\frac{4\pi}{3}}\int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho\sin\phi)^2\rho^2\sin\phi \, d\rho d\phi d\theta \\
&= \frac{3}{4\pi}\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin^3\phi \, d\phi\right)\left(\int_0^1 \rho^4 d\rho\right) \\
&= \frac{3}{4\pi}(2\pi)\left(\int_0^{\pi} \sin\phi(1 - \cos^2\phi)d\phi\right)\left(\frac{1}{5}\right) \\
&= \frac{3}{10}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} \\
&= \frac{2}{5}.
\end{aligned}$$